Complex Representation:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

eg. $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$
$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

use for waves:

$$\psi(x,t) = A\cos(kx-\omega t+\varepsilon) = \mathrm{Re}\left[A e^{i(kx-\omega t+\varepsilon)}\right]$$

often just write as:

$$\psi(x,t) = A e^{i(kx-\omega t+\varepsilon)}$$

with Re[...] understood.

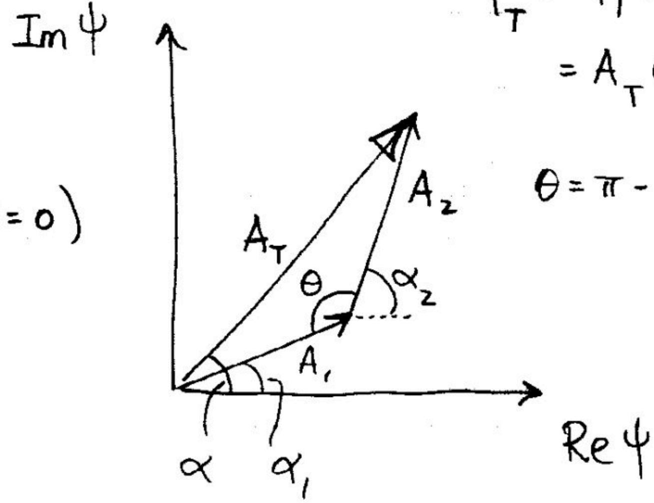
Phasors: add waves of same freq. at fixed point in space.

eg. $\psi_1 = A_1 e^{i(\alpha_1-\omega t)}$; $\alpha_1 = kx_1 + \varepsilon_1$

$\psi_2 = A_2 e^{i(\alpha_2-\omega t)}$; $\alpha_2 = kx_2 + \varepsilon_2$

$$\psi_T = \psi_1 + \psi_2 = A_T e^{i(\alpha-\omega t)}$$

$(t=0)$



$$\theta = \pi - (\alpha_2-\alpha_1)$$

$$A_T^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2-\alpha_1)$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2}$$

• if $E_{01} \neq E_{02}$ (i.e. unequal amp waves) still get beats but I doesn't go to zero at min.

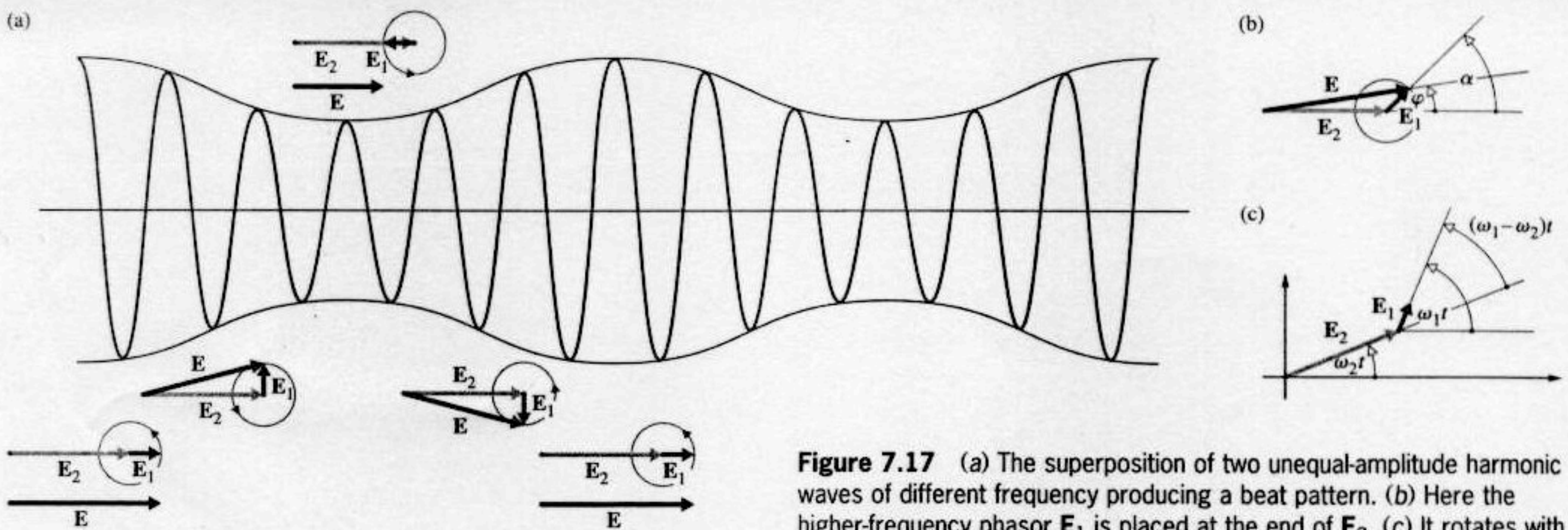


Figure 7.17 (*a*) The superposition of two unequal-amplitude harmonic waves of different frequency producing a beat pattern. (*b*) Here the higher-frequency phasor $\mathbf{E}_1$ is placed at the end of $\mathbf{E}_2$. (*c*) It rotates with the difference frequency.

two waves, diff freq:

$$E_1 = E_{01} \cos(k_1 x - \omega_1 t)$$

$$E_2 = E_{01} \cos(k_2 x - \omega_2 t)$$

$E = E_1 + E_2$ (use $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$)

$$E = 2E_{01}\cos\left[\frac{(k_1+k_2)}{2}x - \frac{(\omega_1+\omega_2)}{2}t\right] \times \cos\left[\frac{(k_1-k_2)}{2}x - \frac{(\omega_1-\omega_2)}{2}t\right]$$

let $\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$ $\omega_m = \frac{1}{2}(\omega_1 - \omega_2)$

$\underbrace{\bar{k} = \frac{1}{2}(k_1 + k_2)}_{\text{average}}$ $\underbrace{k_m = \frac{1}{2}(k_1 - k_2)}_{\text{modulation}}$

$E = \underbrace{2E_{01}\cos(k_m x - \omega_m t)}_{\substack{\text{modulated amp.} \\ \equiv E_0(x,t)}} \cos(\bar{k}x - \bar{\omega}t)$

travelling wave

for $\omega_1 \approx \omega_2$, $\omega_m \ll \bar{\omega}$

$I \propto \langle E(x,t)^2 \rangle \propto E_0^2(x,t)$

$= 4E_{01}^2 \cos^2(k_m x - \omega_m t)$

$= 2E_{01}^2 \left[1 + \cos(2k_m x - 2\omega_m t)\right]$

thus $E_0^2(x,t)$ oscillates about $2E_{01}^2$ with freq $2\omega_m \equiv$ beat freq.

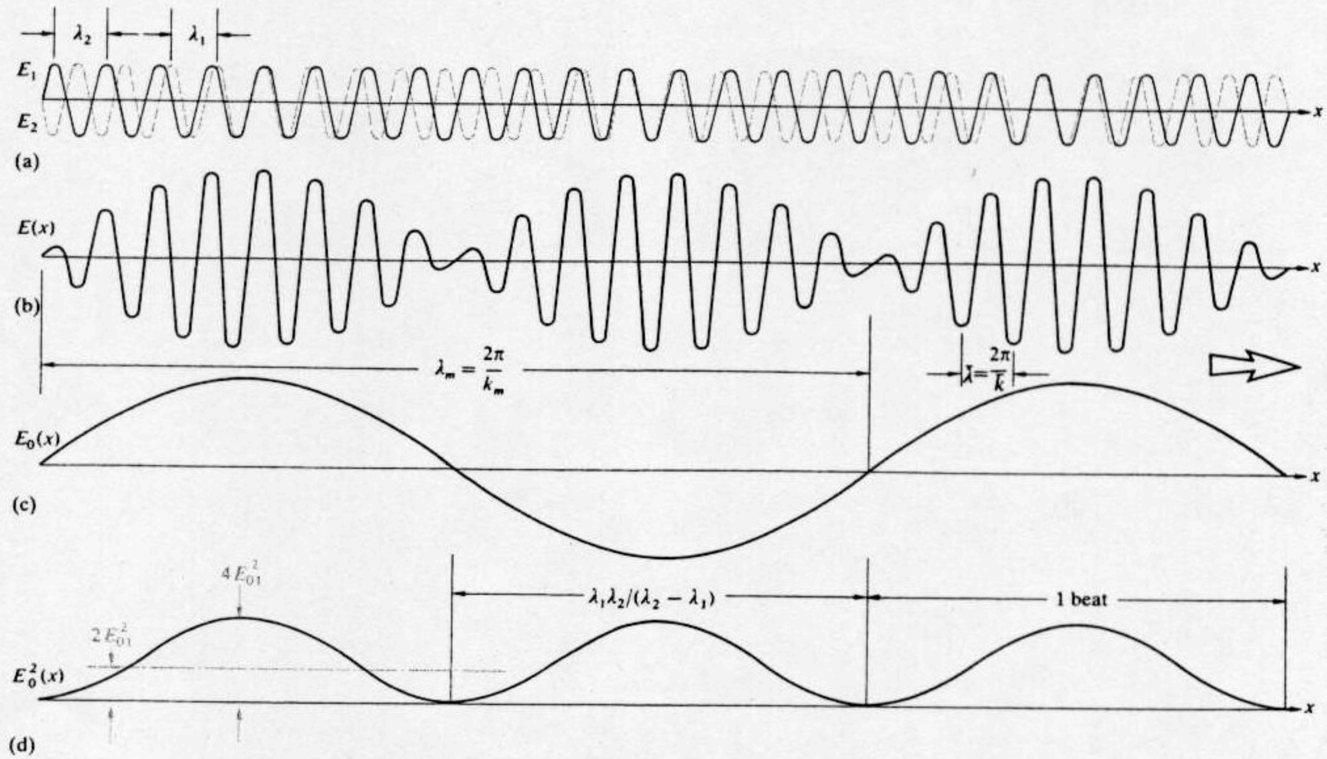


FIGURE 7.13 The superposition of two harmonic waves of different frequency.

Group Velocity

eg. previous example of "carrier wave" at freq. $\bar{\omega}$ with some modulation, Fig. 7.13(b); velocity of peaks, ignoring modulation

$$v = \frac{\bar{\omega}}{k} = \text{phase velocity of carrier}$$

but, also look at velocity of modulation envelope,

group velocity $\quad v_g = \dfrac{\omega_m}{k_m} = \dfrac{\omega_1 - \omega_2}{k_1 - k_2}$

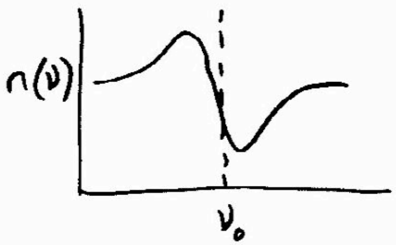
$$= \frac{\Delta\omega}{\Delta k}$$

in general,

$$\boxed{v_g = \left(\frac{d\omega}{dk}\right)_{\bar{\omega}}}$$

eg. in vacuum $\omega = ck$, $\dfrac{d\omega}{dk} = c$ and $v_g = c$, i.e. no pulse broadening.

but, for



(real part of index of refraction)

$$v = c/n = \frac{\omega}{k}$$

$$\Rightarrow \omega = kv$$

$$v_g = \frac{d\omega}{dk} = v + k\frac{dv}{dk}$$

$$= \frac{c}{n} - \frac{kc}{n^2}\frac{dn}{dk}$$

$$= v\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

$$\frac{dv}{dk} = -\frac{c}{n^2}\frac{dn}{dk}$$

"normal" $dn/d\nu > 0$

$\Rightarrow dn/dk > 0$

since $\nu = k \cdot 2\pi v$

$\Rightarrow v_g < v$

or, in terms of $\lambda = \dfrac{2\pi}{k}$

$$\frac{dn}{dk} = \frac{dn}{d\lambda}\frac{d\lambda}{dk} = -\frac{2\pi}{k^2}\frac{dn}{d\lambda}$$

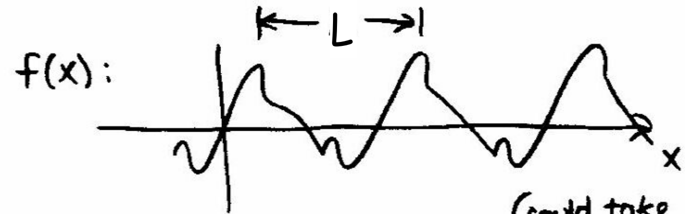
$$v_g = v\left(1 + \frac{2\pi}{kn}\frac{dn}{d\lambda}\right)$$

$$= v\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right).$$

normally $dn/d\lambda < 0$ ✓

Fourier Series

- form periodic waveform from sum of harmonic waves



(could take $x \to x - vt$)

- decompose into harmonic components, and look at propagation for each.

write

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx$$

with $k = \frac{2\pi}{L}$

then,
$$A_m = \frac{2}{L}\int_0^L f(x)\cos mkx\,dx$$

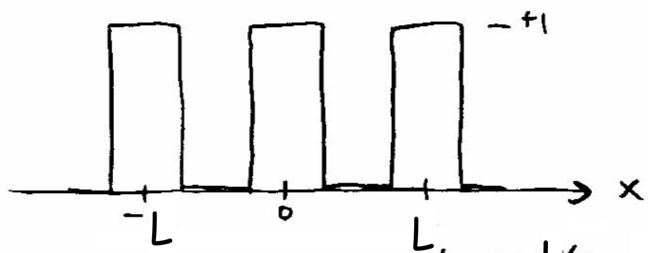
$$B_m = \frac{2}{L}\int_0^L f(x)\sin mkx\,dx$$

even $f(x) \Rightarrow B_m \equiv 0$; odd $f(x) \Rightarrow A_m \equiv 0$

example - square wave.

http://www.earlevel.com/Digital%20Audio/harmonigraf.html

Square Wave:

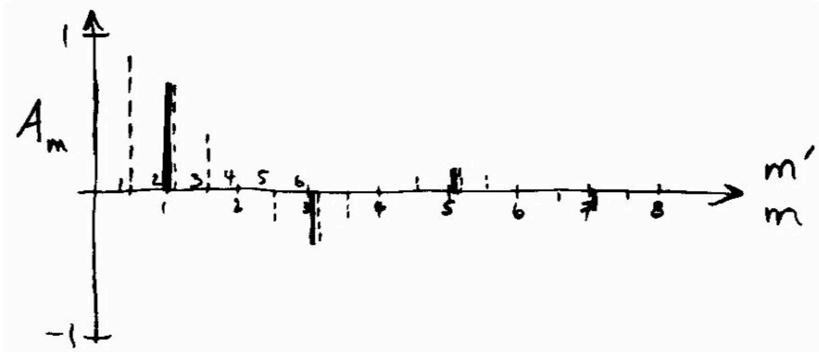


for $0<x<\lambda$, $f(x) = \begin{cases} +1 & \begin{cases} 0<x<L/4 \\ 3L/4<x<L \end{cases} \\ 0 & \text{elsewhere} \end{cases}$

$f(x)$ even $\Rightarrow B_m \equiv 0$

$$A_m = \frac{2}{L}\int_0^L f(x)\cos mkx\,dx$$
$$= \frac{2}{L}\left[\int_0^{L/4}\cos mkx\,dx + \int_{3L/4}^{L}\cos mkx\,dx\right]$$
$$= \frac{2}{L}\frac{1}{mk}\left[\sin mkx\Big|_0^{L/4} + \sin mkx\Big|_{3L/4}^{L}\right]$$
$$= \frac{1}{m\pi}\Big[\sin m\frac{\pi}{2} - 0 + 0 - \underbrace{\sin m\frac{3\pi}{2}}_{-\sin m\frac{\pi}{2}}\Big]$$
$$= \frac{2}{m\pi}\sin m\frac{\pi}{2}$$

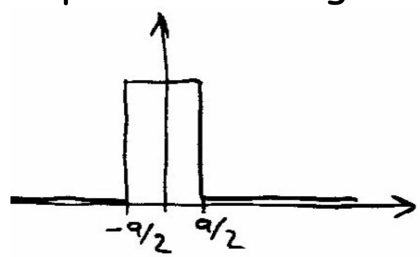
what if have:



then

$$A_{m'} = \frac{2}{m'\pi}\left[\sin m'\frac{\pi}{4} - \sin m'\frac{7\pi}{4}\right] = \frac{4}{m'\pi}\sin m'\frac{\pi}{4}$$

if continue to shrink width of pulse, eventually get single pulse at origin:
(don't consider delta-fcn, but rather, single pulse with large separation to next pulse)



$L \to \infty$, $k \to 0$, get very dense spacing of mk values in freq. (ie. spatial freq) space.

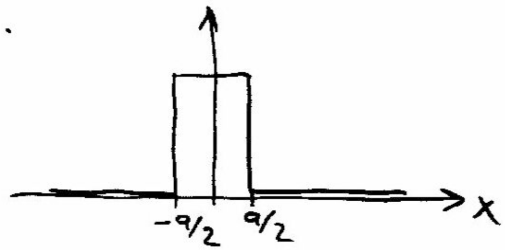
then, use continuous Fourier transform:

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(k)\cos kx\,dk + \int_0^\infty B(k)\sin kx\,dk\right]$$

with $A(k) = \int_{-\infty}^{\infty} f(x)\cos kx\,dx$

$$B(k) = \int_{-\infty}^{\infty} f(x)\sin kx\,dx$$

eg.

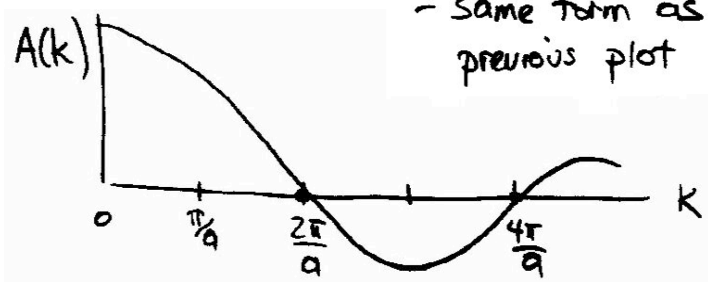


$B(k) \equiv 0$

$$A(k) = \int_{-\infty}^{\infty} f(x)\cos kx\,dx = \int_{-a/2}^{a/2} \cos kx\,dx$$

$$= \frac{1}{k}\sin kx\Big|_{-a/2}^{a/2} = \frac{1}{k}\left(\sin\frac{ka}{2} - \sin\frac{-ka}{2}\right)$$

$$= \underline{\underline{\frac{2}{k}\sin\frac{ka}{2}}}$$



- same form as previous plot

<u>complex form:</u>

<u>periodic case</u>, $0 < x < \lambda$

$$f(x) = \sum_{m=-\infty}^{\infty} A_m e^{imkx} \qquad \text{with } k = \frac{2\pi}{\lambda}$$

derive A_m: multiply by $e^{-im'kx}$ and integrate over x:

$$\int_0^{\lambda} f(x) e^{-im'kx}\,dx = \int_0^{\lambda} dx\, e^{-im'kx} \sum_{m=-\infty}^{\infty} A_m e^{imkx}\,dx$$

$$= \sum_{m=-\infty}^{\infty} A_m \int_0^{\lambda} \underbrace{e^{i(m-m')kx}\,dx}_{\delta_{mm'}\lambda} = \lambda A_{m'}$$

thus

$$\underline{A_m = \frac{1}{\lambda}\int_0^{\lambda} f(x) e^{-imkx}\,dx}$$

<u>nonperiodic</u>, $-\infty < x < \infty$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k) e^{ikx}\,dk$$

multiply by $e^{-ik'x}$ and integrate over x:

$$\int_{-\infty}^{\infty} f(x) e^{-ik'x}\,dx = \int_{-\infty}^{\infty} dx\, e^{-ik'x} \int_{-\infty}^{\infty} A(k) e^{ikx}\,dk \frac{1}{2\pi}$$

$$= \int_{-\infty}^{\infty} dk\, A(k) \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k-k')x}}_{2\pi\delta(k-k')} \frac{1}{2\pi} = A(k')$$

thus

$$\underline{A(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx}$$